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G.A.G.A. affine by polynomial growth

by

Hakuki Yamaguchi

§0. Introduction

P.Deligne and G.Maltsiniotis studied G.A.G.A. on a non-singular separated scheme of finite type over $\mathbf{C}$. Their method is dependent on L^2 estimates for the $\bar{\partial}$ operator by Hörmander. On the other hand, using the Čech theory and Cousin integrals, N.Sasakura studied a polynomial growth cochain complex for a p.g. coherent sheaf. We shall define a polynomial growth cochain complex for a locally free sheaf(i.e.a vector bundle)by his techniques in [2] and give G.A.G.A. similar to the one due to [1]. This is a generalization of the theorem(3.4) in [2].

The contents of the various sections are as follows. In §1 we summarise some results in [2], which we shall use in the last section. In §2 we shall define a polynomial growth cochain complex for locally free sheaf over a smooth affine variety over $\mathbf{C}$. In §3 we shall give G.A.G.A.(Th.(3.0)) for a vector bundle over a smooth affine variety.

The idea of this paper is much in debt to Professor N.Sasakura. Professor T.Kori has encouriged me in my studies. I wish to thank them.

§1. Preliminaries

Let X be a smooth affine variety over $\mathbf{C}$. Unless we say otherwise, we regard X as the analytic variety (with the underlying analytic structure). Moreover, $\mathcal{O}_X$ = the structure sheaf of the analytic variety. Let $X \overset{i}{\hookrightarrow} \mathbf{C}^N$ be an embedding of X and $g = |z| + 1$ a growth function increasing at infinity, where z is the coordinate of $\mathbf{C}^N$.

Definition(1.0)

By a polynomial growth (or simply p.g.) covering of X of size $\sigma=(\sigma_1,\sigma_2)(\sigma_1 \geqq 1, \sigma_2 \geqq 1)$ with respect to g, we mean the collection as follows:

$\hat{\mathcal{A}}_\sigma(X;g)$(or simply,$=\hat{\mathcal{A}}_\sigma(X)$) $=\{\tilde{U}_\sigma(Q;g)=X\cap U_\sigma(Q;g);\ Q\in X\}$, where $U_\sigma(Q;g)$ is the disc,whose center is Q and radius is $\frac{1}{\sigma_1}g(Q)^{-\sigma_2}$.

Definition(1.1)

By a p.g. cochain compex of $\mathcal{O}_X^k$ for $\hat{\mathcal{A}}_\sigma(X)$,we mean the subgroup of the cochain compex $C^*(\hat{\mathcal{A}}_\sigma(X),\mathcal{O}_X^k)$ as follows:

$C^*(\hat{\mathcal{A}}_\sigma(X),\mathcal{O}_X^k;g)_{p.g}=\{\varphi\in C^*(\hat{\mathcal{A}}_\sigma(X),\mathcal{O}_X^k);\exists\alpha,\ \beta\geqq 1,|\varphi|\leqq\alpha(|z|+1)^\beta$,where $|\varphi|$ runs through all the components of $\varphi\}$

,where $\mathcal{O}_X^k$ is a direct sum of $\mathcal{O}_X$.

Theorem(1.2)

Let X be as above. Then $H^q(\varinjlim_\sigma C^*(\hat{\mathcal{A}}_\sigma(X),\mathcal{O}_X^k)_{p.g})=0$ for $q\geqq 1$

$$H^0(\varinjlim_\sigma C^*(\hat{\mathcal{A}}_\sigma(X),\mathcal{O}_X^k)_{p.g})=\Gamma(X_{alg.}\mathcal{O}_{X_{alg.}}^k)$$

,where $\mathcal{O}_{X_{alg.}}$ is a sructure sheaf as an algebraic variety.

We shall extend it to a locally free sheaf in the last section.

2. Polynomial growth cochain complexes for locally free sheaves.

(i) Let X, $\mathcal{O}_X$ and g be the same as those in §1.

Proposition(2.1)

Let $X \overset{i'}{\hookrightarrow} C^{N'}$ be another embedding of X and $g' = |z'| + 1$ a growth function, Where z' is the coordinate of $C^{N'}$. Then

$$\varinjlim_{\sigma} C^*(\hat{\mathcal{R}}_\sigma(X), \mathcal{O}_X{}^k; g)_{p.g} = \varinjlim_{\sigma} C^*(\hat{\mathcal{R}}_\sigma(X), \mathcal{O}_X{}^k; g')_{p.g}.$$

In order to prove this proposition we need the following Lemma.

Lemma(2.2)

Let $H(z)$ be a polynomial of $z = (z_1 \cdots\cdots z_N)$, then there exist $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta_1 > 0$, $\beta_2 > 0$ such that

(*) $\quad \alpha_1(|z| + 1)^{\alpha_2} \leqq |H(z)| + 1 \leqq \beta_1(|z| + 1)^{\beta_2}.$

proof of (2.2)

We may write $H(z) = \sum_{|\nu| \leq n} a_\nu z_1^{\nu_1} \cdots\cdots z_N^{\nu_N}$. Then we have

(a) $\quad |H(z)| \leqq \sum_{|\nu| \leq n} |a_\nu z_1^{\nu_1} \cdots\cdots z_N^{\nu_N}| \leqq \max_{|\nu| \leq n} |a_\nu| \cdot (|z| + 1)^n.$

We derive the inequality of the right hand side in (*) from (a).

On the other hand we divide the numerator and the denominator of the following function by $|z|^n$,

(b) $$\frac{(|z| + 1)^n}{\left|\sum_{|\nu| \leq n} a_\nu z_1^{\nu_1} \cdots z_N^{\nu_N}\right| + 1}$$

And $|z| \longrightarrow \infty$, then (b) has a non-zero limit, so that there exist $r > 0$ and $M' > 0$ such that

(c) $\quad (|z| + 1)^n \leqq M'(|H(z)| + 1) \quad$ for $|z| > r$.

Now we have

(d) $\quad (|z| + 1)^n \leqq (r + 1)^n \quad$ for $|z| \leqq r$.

Then we get the following inequality by (c) and (d):

(e) $\frac{1}{M}(|z| + 1)^n \leqq |H(z)| + 1$, where $M = \mathrm{Max}\ ((r + 1)^n, M')$.

Q.E.D

proof of (2.1)

By the assumption of the embeding of X there exist polynomials $f_i(z) \in \mathbf{C}[z_1, \cdots, z_N]$ (i = 1 $\cdots\cdots$ N) such that

(a) $\quad z_i' = f_i(z) \qquad (i = 1 \cdots\cdots N').$

From (a) and Lemma(2.2) we have

(b) $\quad \alpha_1(|z| + 1)^{\alpha_2} \leqq |z'| + 1 \leqq \beta_1(|z| + 1)^{\beta_2}$ on X.

We take an element $\varphi \in C^*(\hat{\mathcal{A}}_\sigma(X), \mathcal{O}_X{}^k; g)_{p.g}$. By the definition of $C^*(\hat{\mathcal{A}}_\sigma(X), \mathcal{O}_X{}^k; g)_{p.g}$ and (b) we get

(c) $\quad |\varphi| = \gamma_1(|z'| + 1)^{\gamma_2}$.

(c) shows that φ is contained in $C^*(\hat{\mathcal{A}}_\delta(X), \mathcal{O}_X{}^k; g')_{p.g}$.

Q.E.D.

Prop.(2.1) shows that p.g cochain complex is independent of any embedding of X.

Let E be an algebraic vector bundle over X and $X = \bigcup_{i \in I} X_{f_i}$ (= $\bigcup_{i \in I} X_i$ simply) an affine open covering, where E is trivial over each X_i and f_i $(i \in I)$ is a polynomial. Taking a subset J of I, X_J denotes $X_{f_1} \cap \cdots \cap X_{f_s} = X_{f_1 \cdots f_s}$. There is a wellknown diagram:

$$(*)\quad \begin{array}{ccc} C^N - \{f_1 \cdots f_s = 0\} & \longrightarrow & S = \{1 - f_1(z)w_1 = 0 \ \cdots \ 1 - f_s(z)w_s = 0\} \subset C^N \times C^s \\ (z) & \longmapsto & (z) \times \left(\frac{1}{f_1(z)}, \cdots\cdots, \frac{1}{f_s(z)}\right) \\ \cup & & \cup \\ X_J & \longrightarrow & \widetilde{X}_J = S \cap \pi^{-1}(X_J) \end{array}$$

, where π ; $C^N \times C^s \longrightarrow C^N$ is a projection and (w) is the coordinate of C^s. Since X_J and $\widetilde{X}_J$ are isomorphic each other in (*),

$g_J = 1 + |z| + \frac{1}{|f_1(z)|} + \cdots\cdots + \frac{1}{|f_s(z)|}$ is a growth function on X_J. E is trivial over X_J and we can regard E as an analytic vector bundle, so

(**) $\qquad E|X_J \cong \mathcal{O}_X^k | X_J.$

From (**) we can define the p.g. cochain complex $C^*(\hat{\mathcal{A}}_\sigma(X_J;g_J),E)_{p.g}$. This p.g. cochain complex is independent of the way of taking any isomorphism in (**) and uniquely determined by Prop.(2.1).

Proposition(2.3)

Let K be a subset of J, then $X_J \subset X_K$ and there is a restriction map:

$$res_{KJ};\ C^*(\hat{\mathcal{A}}_\sigma(X_J;g_J),E)_{p.g} \longrightarrow C^*(\hat{\mathcal{A}}_\sigma(X_K;g_K),E)_{p.g}.$$

proof

Because

$$\begin{cases} g_K(z) = 1 + |z| + \frac{1}{|f_1(z)|} + \cdots\cdots + \frac{1}{|f_s(z)|} \\ g_J(z) = 1 + |z| + \frac{1}{|f_1(z)|} + \cdots\cdots + \frac{1}{|f_s(z)|} + \quad + \frac{1}{f_t(z)}, \end{cases}$$

we have $g_K(Q) \leqq g_J(Q)$ for $\forall Q \in X_J$. This means that

(a) $U_\sigma(Q;g_J) \supset U_\sigma(Q;g_K)$, where $U_\sigma(Q;g_J)$ is the disc, whose center is Q and radius is $\frac{1}{\sigma_1} g_J^{-\sigma_2}$. We have the following refining map by (a):

$$\text{(b)}\quad r\ ;\ \begin{array}{ccc} \hat{\mathcal{A}}_\sigma(X_K;g_K) & \longrightarrow & \hat{\mathcal{A}}_\sigma(X_J;g_J) \\ \cup\!\!\!\!\!\!\smile & & \cup\!\!\!\!\!\!\smile \end{array}$$

$$\tilde{U}_\sigma(Q;g_K) = U_\sigma(Q;g_K) \cap X_K \longmapsto \tilde{U}_\sigma(Q;g_J) = U_\sigma(Q;g_J) \cap X_J.$$

(b) induces the following restriction map:

$$res_{KJ};\ C^*(\hat{\mathcal{A}}_\sigma(X_J;g_J),E)_{p.g} \longrightarrow C^*(\hat{\mathcal{A}}_\sigma(X_K;g_K),E)_{p.g}.$$

Q.E.D.

(ii) Let X, E and g_J be as above. We make a double complex in the following manner:

$$K^{pq}_{\sigma,p.g} \Leftarrow \overset{\text{alt.}}{\bigoplus_{|J|=q+1}} C^p(\hat{\mathcal{A}}_\sigma(X_J;g_J),E)_{p.g}.$$

Differentials δ_1 and δ_2 can be defined asfollows:

$$\begin{array}{ccc} \delta_1;\ K^{pq}_{\sigma,p.g} & \longrightarrow & K^{p+1q}_{\sigma,p.g} \\ \Psi & & \Psi \\ \varphi & \longmapsto & \delta_1\varphi \end{array}$$

where $(\delta_1\varphi)_{j_0\cdots j_{p+1}} = \sum_{k=0}^{p+1}(-1)^k \mathrm{res}(\varphi_{j_0\cdots\hat{j}_k\cdots j_{p+1}})$ and $\overset{\text{alt.}}{\bigoplus}$ denotes an alternating sum.

$$\begin{array}{ccc} \delta_2;\ K^{pq}_{\sigma,p.g} & \longrightarrow & K^{pq+1}_{\sigma,p.g} \\ \Psi & & \Psi \\ \varphi & \longmapsto & \delta_2\varphi \end{array}$$

, where $(\delta_2\varphi)_{i_0\cdots i_{q+1}} = \sum_{k=0}^{q+1}(-1)^k \mathrm{res}_{(i_0\cdots i_{q+1})(i_0\cdots\hat{i}_k\cdots i_{q+1})}(\varphi_{i_0\cdots\hat{i}_k\cdots i_{q+1}})$

and $\mathrm{res}_{(i_0\cdots i_{q+1})(i_0\cdots\hat{i}_k\cdots i_{q+1})}$ is the restriction map in Prop.(2.3). On the other hand we have the following refining map:

$$\begin{array}{lccc} \text{(a)} & r_i;\ \hat{\mathcal{A}}_\sigma(X_i;g_i) & \longrightarrow & \mathcal{A}_\sigma(X;g) \\ & \Psi & & \Psi \\ & X_i \cap U_\sigma(Q;g_i) & \hookrightarrow & X \cap U_\sigma(Q;g), \end{array}$$

where $g_i = 1 + |z| + \frac{1}{|f_i(z)|}$ and $g = 1 + |z|$. From (a) we have the following diagram:

$$\begin{array}{ccc} C^p(\hat{\mathcal{A}}_\sigma(X;g),E) & \xrightarrow{r=\oplus r_i} & \bigoplus_i C^p(\hat{\mathcal{A}}_\sigma(X_i;g_i),E) \\ & & \Big\downarrow \delta_2 \\ & & \overset{\text{alt.}}{\bigoplus_{i_0i_1}} C^p(\hat{\mathcal{A}}_\sigma(X_{i_0i_1};g_{i_0i_1}),E) \end{array}$$

, where $C^p(\hat{\mathcal{A}}_\sigma(X;g),E)$, $C^p(\hat{\mathcal{A}}_\sigma(X_i;g_i),E)$ and $C^p(\hat{\mathcal{A}}_\sigma(X_{i_0i_1};g_{i_0i_1}),E)$ are the usual cochain complexes and

$$g_{i_0i_1} = 1 + |z| + \frac{1}{|f_{i_0}(z)|} + \frac{1}{|f_{i_1}(z)|}.$$

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Proposition(2.3)

We take the inductive limit of the above diagram:

$$\begin{array}{ccc} \varinjlim_{\sigma} C^p(\hat{\mathcal{A}}_\sigma(X),E) & \xrightarrow{r=\oplus r_i} & \bigoplus_i \varinjlim_{\sigma} C^p(\hat{\mathcal{A}}_\sigma(X_i),E) \\ & & \Big\downarrow \delta_2 \\ & & \overset{\text{alt.}}{\bigoplus_{i_0 i_1}} \varinjlim C^p(\hat{\mathcal{A}}_\sigma(X_{i_0 i_1}),E) \end{array}$$

Then Ker δ_2 = Im r .

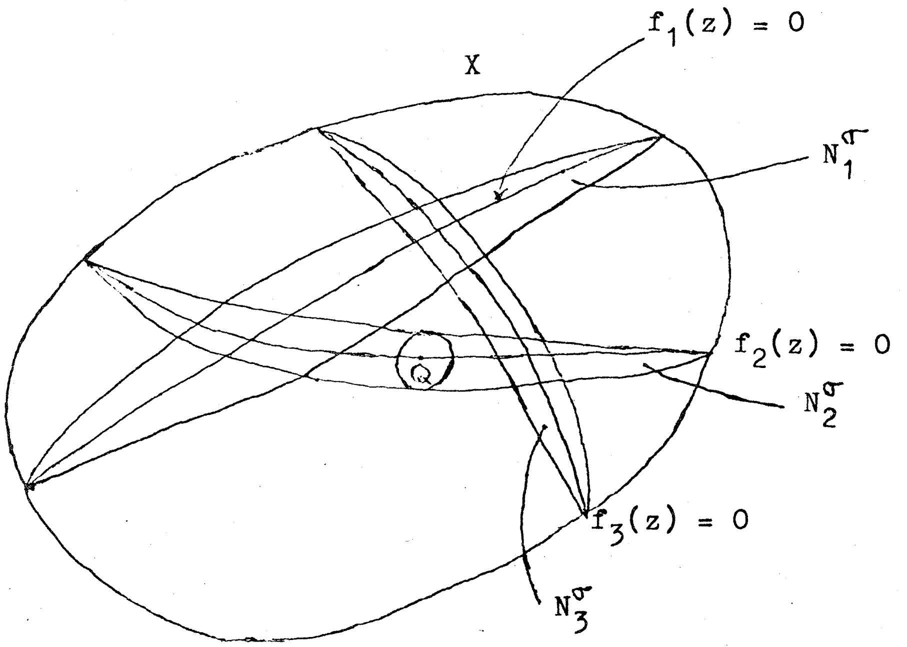
proof

Ker $\delta_2 \supset$ Im r is trivial.

First we show that Ker $\delta_2 \subset$ Im r in case p=0. We take an element $\varphi = (\varphi_i) \in \bigoplus_i C^0(\hat{\mathcal{A}}_\sigma(X_i),E)$. If we can take $\sigma' \geq \sigma$ $(\sigma_1' \geqq \sigma_1\ ,\ \sigma_2' \geqq \sigma_2)$ which is independent of $Q \in X$ such that there exists $i = i_Q \in I$ for $\forall Q \in X$ and (a) $\tilde{U}_{\sigma'}(Q;g) \subset \tilde{U}_\sigma(Q;g_i)$, we may take $\Psi = \{\Psi_Q\}_Q \in C^0(\hat{\mathcal{A}}_\sigma(X),E)$ such that $\Psi_Q = \varphi_{i_Q} | \tilde{U}_{\sigma'}(Q;g)$. On the other hand for another correspondence $j = j_Q$ as above we can define Ψ'_Q as $\varphi_{j_Q} | \tilde{U}_{\sigma'}(Q;g_j)$, then there exists $i \in I$ such that $\varphi_{i_Q} = \varphi_{j_Q}$ on $\tilde{U}_\sigma(X_{ij};g_{ij})$ which is a cocycle condition (i.e. $\delta_2\varphi = 0$). This means that $\Psi'_Q = \Psi_Q$. Therefore the $\{\Psi_Q\}_Q \in C^0(\hat{\mathcal{A}}_\sigma(X),E)$ is independent of the above correspondence, $X \ni Q \longmapsto i = i_Q \in I$. Then we shall define the following manner:

$$X \ni Q \longrightarrow i_Q \in I,$$

where this i_Q is i such that $g_i(Q) = \min_{j \in I}\{g_j(Q)\}$. Since I is finite, we can choose it. The existence of $\sigma' \geqq \sigma$ which is independent of $Q \in X$ is derived from the following figure:



, where $N_i^\sigma = \bigcup_{Q \in \{f_i = 0\}} U_0(Q;g)$ and $g = 1 + |z|$. When we choose sufficiently large σ, $X = \bigcup_{i \in I} X - N_i^\sigma$. We can prove $\mathrm{Ker}\, \delta_2 \subset \mathrm{Im}\, r$ for the p-th cochain complex $C^p(\hat{\mathcal{A}}_\sigma(X), E)$ in the same manner.

Q.E.D.

Now we can define a p.g. cochain complex $C^*(X,E)_{p.g}$ for X.

Definition(2.4)

$$C^p(X,E)_{p.g} = \varinjlim_\sigma (\mathrm{Im}\, r)^{p,\sigma}_{p.g},$$

where $(\mathrm{Im}\, r)^{p,\sigma}_{p.g} \subset \bigoplus_i C^p(\hat{\mathcal{A}}_\sigma(X_i;g_i),E)_{p.g}$. Since $(\mathrm{Im}\, r)^{p,\sigma} \subset \bigoplus_i C^p(\hat{\mathcal{A}}_\sigma(X_i;g_i),E)$ and $C^p(\hat{\mathcal{A}}_\sigma(X_i;g_i),E)_{p.g}$ has defined, we can attach $(\mathrm{Im}\, r)^{p,\sigma}$ to a p.g. condition.

Remark(2.5)

As r is injective, Im r is uniquely defined. (i.e. $C^p(X,E)_{p.g}$ is independent of an affine open covering $X = \bigcup_{i \in I} X_i$.)

Proposition(2.6)

$C^*(X,E)_{p.g}$ is a complex in the usual manner.

This is trivial by the definition.

§3. The comparison and vanishing theorem

Let X and E be as above and $C^*(X,E)_{p.g}$ the one in the def.(2.4).

Theorem(3.0)

$$H^q(C^*(X,E)_{p.g}) = 0 \quad \text{for} \quad q \geqq 1,$$
$$H^0(C^*(X,E)_{p.g}) \cong \Gamma(X_{alg.}, E_{alg.}).$$

When we regard E as an algebraic vector bundle, we write $E_{alg.}$.

In order to prove the Th.(3.0) we think the following diagram;

(*)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & C^0(\mathfrak{U}, E_{alg.}) \arrow[r] \arrow[d, "i_1"] & C^1(\mathfrak{U}, E_{alg.}) \arrow[r] \arrow[d, "i_1"] & {} \\
0 \arrow[r] & C^0(X,E)_{p.g} \arrow[r, "i_2"] \arrow[d] & K^{00}_{p.g} \arrow[r] \arrow[d, "\delta_1"] & K^{01}_{p.g} \arrow[r] \arrow[d, "\delta_2"] & {} \\
0 \arrow[r] & C^1(X,E)_{p.g} \arrow[r, "i_2"] & K^{10}_{p.g} \arrow[r] & K^{11}_{p.g} \arrow[r] & {} \\
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & C^p(X,E)_{p.g} \arrow[r, "i_2"] \arrow[d] & K^{p0}_{p.g} \arrow[r] \arrow[d, "\delta_1"] & K^{p1}_{p.g} \arrow[r] \arrow[d, "\delta_2"] & {} \\
 & {} & {} & {} &
\end{tikzcd}$$

, where $\mathcal{U} = \{X_i\}_{i\in I}$ is an affine open covering and

$$K^{pq}_{p.g} = \varinjlim_{\sigma} K^{pq}_{\sigma,p.g} .$$

By the simple observation we have the following:

$$\text{(a)}\ K^{0q}_{p.g} = \varinjlim_{\sigma} \overset{\text{alt.}}{\bigoplus_{i_0\cdots i_q}} C^0(\hat{\mathcal{A}}_\sigma(X_{i_0\cdots i_q}),E)_{p.g} \hookrightarrow \overset{\text{alt.}}{\bigoplus_{i_0\cdots i_q}} \Gamma(X_{i_0\cdots i_q},E)_{p.g}$$

$$\text{(b)}\ \overset{\text{alt.}}{\bigoplus_{i_0\cdots i_q}} \Gamma(X_{i_0\cdots i_q},E)_{p.g} = \overset{\text{alt.}}{\bigoplus_{i_0\ i_q}} \Gamma(X_{i_0\ i_q},E_{alg.})$$

$$\cong C^q(\mathcal{U},E_{alg.}) .$$

We derive the first equality in (b) from the Sasakura's th.(1.2) and the second equality in (b) from the Serre's vanishing theorem of affine varieties. Therefore i_1 is injective from (a) and (b). Also i_2 is injective by Prop.(2.3). On the other hand

$$H^p(K^{*q}_{p.g}) = \overset{\text{alt.}}{\bigoplus_{i_0\cdots i_q}} \varinjlim_{\sigma} H^p(C^*(\hat{\mathcal{A}}_\sigma(X_{i_0\cdots i_q}),E)_{p.g} = 0 \quad \text{for} \quad p \geqq 1 .$$

by the Sasakura's vanishing theorem(1.2). This means that (1) in the following proposition is true.

<u>Proposition</u>(3.1)

The complexes,

$$(1)\ K^{0q}_{p.g} \longrightarrow K^{1q}_{p.g} \longrightarrow \cdots\cdots \longrightarrow K^{pq}_{p.g} \longrightarrow \cdots \qquad \text{for } q \geqq 0$$

$$(2)\ K^{p0}_{p.g} \longrightarrow K^{p1}_{p.g} \longrightarrow \cdots\cdots\cdots \longrightarrow K^{pq}_{p.g} \longrightarrow \cdots \qquad \text{for } p \geqq 0,$$

are exact.

We must prove (2).

<u>Lemma</u>(3.2)

Let $\mathcal{U} = \{X_i\}_{i\in I}$ be as above. Then $H^1(K^{p*}_{p.g}) = 0$ for $p \geqq 0$.

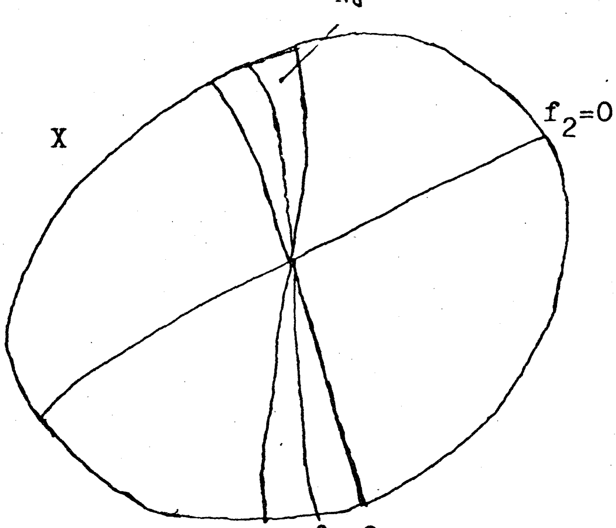
proof of Lemma(3.2)

We shall prove it by the induction on the number of sheets of the covering $\mathcal{U}$.

(i) When $\#I = 2$, $\mathcal{U} = \{X_1, X_2\}$.

We define a neighborhood N_σ of $\{f_1=0\}$ in the following manner:

$N_\sigma = \bigcup_{Q \in V_1 - V_{12}} U_\sigma(Q; d(Q,V_{12}))$, where $V_1 = \{f_1=0\}$ and $V_{12} = \{f_1=0\} \cap \{f_2=0\}$ and $d(Q,V_{12})$ is a distance from Q to V_{12} and $U_\sigma(Q;d(Q,V_{12}))$ is the disc whose center is Q and radius $\frac{1}{\sigma_1} d(Q,V_{12})^{+\sigma_2}$. Then we have the following figure:



We take an element $\varphi_{12} \in C^p(\hat{\mathcal{A}}_\sigma(X_{12}),E)_{p.g}$, then we give φ_1 and φ_2 in the following:

$$\varphi_1 \begin{cases} = -\varphi_{12} & \text{on } N_\sigma \\ = 0 & \text{on } X-N_\sigma \end{cases}$$

$$\varphi_2 \begin{cases} = 0 & \text{on } N_\sigma \\ = \varphi_{12} & \text{on } X-N_\sigma. \end{cases}$$

Then we have $\varphi_{12} = \varphi_2 - \varphi_1$. When we take sufficiently large $\sigma' (\geqq \sigma)$, $\varphi_i \in C^p(\hat{\mathcal{A}}_{\sigma'}(X_i),E)_{p.g}(i=1,2)$ by the Lojasiewicz inequality and the triangle inequality. Therefore $H^1(K^{p*}_{p.g}) = 0$ for $\forall p \geqq 0$.

(ii) When $\#I=3$, $\mathfrak{N}=\{X_1,X_2,X_3\}$ and we take

$$\varphi=(\varphi_{12},\varphi_{13},\varphi_{23})\in\overset{\text{alt.}}{\underset{i,j}{\oplus}} C^p(\hat{\mathcal{A}}_\sigma(X_{ij}),E)_{p.g}.$$

Applying the assumption of the induction to $\{X_1,X_2\}$, there exist $\varphi_1\in C^p(\hat{\mathcal{A}}_\sigma(X_1),E)_{p.g}$ and $\varphi_2\in C^p(\hat{\mathcal{A}}_\sigma(X_2),E)_{p.g}$ such that $\varphi_{12}=\varphi_2-\varphi_1$. By the cocycle condition, $\varphi_{23}-\varphi_{13}+\varphi_{12}=0$, we have $(\varphi_{23}+\varphi_2)-(\varphi_{13}+\varphi_1)=0$. We define newly as follows:

$$\tilde{\varphi}_{23}=\varphi_{23}-\varphi_2\ ,\quad \tilde{\varphi}_{13}=\varphi_{13}-\varphi_1\ ,\quad \tilde{\varphi}_{12}=0\ .$$

Since $(\tilde{\varphi}_{13},\tilde{\varphi}_{23})$ is a co-cycle for (X_{13},X_{23}), we can define $\tilde{\varphi}_3$ as follows:

$$\tilde{\varphi}_3\begin{cases}=\varphi_{13} & \text{on } X_{13}\\ =\varphi_{23} & \text{on } X_{23}\\ =0 & \text{on } X_3-(X_{13}\cup X_{23}).\end{cases}$$

This cochain $\tilde{\varphi}_3$ is not contained in $C^p(\hat{\mathcal{A}}_\sigma(X_3),E)_{p.g}$. Now we define $\varphi_3\in C^p(\hat{\mathcal{A}}_\sigma(X_3),E)_{p.g}$ in the following manner:

$$\varphi_3\begin{cases}=\tilde{\varphi}_3 & \text{on } X_3-N\\ =0 & \text{on } N\end{cases}$$

$$\varphi_1\begin{cases}=\tilde{\varphi}_3 & \text{on } N-X_1^c\\ =0 & \text{on otherwise}\end{cases}$$

$$\varphi_2\begin{cases}=\tilde{\varphi}_3 & \text{on } N-X_3^c\\ =0 & \text{otherwise}\end{cases}$$

, where X_i^c is a complement of X_i. Then $\varphi_3\in C^p(\hat{\mathcal{A}}_\sigma(X_3),E)_{p.g}$ and $H^1(K^{p*}_{p.g})=0$ for $p\geqq 0$.

(iii) When $\#I=n$, we can show that $H^1(K^{p*}_{p.g})=0$ for $p\geqq 0$ in the same manner as (ii).

<u>Q.E.D.</u>

Let $\mathfrak{N}^n=\{X_1,\ldots\ldots,X_n\}$ be a covering of X and $K^{p*}_{p.g}(\mathfrak{N}^n)$ as $K^{p*}_{p.g}$ for $\mathfrak{N}^n$.

Lemma(3.3)

We assume that $H^i(K^{p*}_{p.g}(\mathcal{N}^n)) = 0$ for $i = q-1,\ q\ (q-1 \geqq 1)$.
Then $H^q(K^{p*}_{p.g}(\mathcal{N}^{n+1})) = 0$.

proof

We take an element $\varphi_{n+1} \in Z^q(K^{p*}_{p.g}(\mathcal{N}^{n+1}))$. We restrict φ_{n+1} $\mathcal{N}^n$ and $\varphi_n \in Z^q(K^{p*}_{p.g}(\mathcal{N}^n))$ denotes it. By the assumption of the induction there is $\Psi_n \in K^{pq-1}_{p.g}(\mathcal{N}^n)$ such that $\delta_2 \Psi_n = \varphi_n$. Now we define $\Psi_{n+1} \in K^{pq-1}_{p.g}(\mathcal{N}^{n+1})$ as follows:

$$\begin{cases} \Psi_{n+1}(X_{i_1} \cap \cdots \cap X_{i_q}) = \Psi_n(X_{i_1} \cap \cdots \cap X_{i_q}) (1 \leqq i_1 < \cdots < i_q \leqq n) \\ \Psi_{n+1}(X_{i_1} \cap \cdots \cap X_{i_q}) = 0 \qquad (i_q = n+1). \end{cases}$$

We define an element $\varphi'_{n+1} \in Z^q(K^{p*}_{p.g}(\mathcal{N}^{n+1}))$ by

$$(*)\quad \varphi'_{n+1} = \varphi_{n+1} - \delta_2 \Psi_{n+1} \begin{cases} = 0 & \text{on } X_{i_1} \cap \cdots \cap X_{i_{q+1}} \ (i_{q+1} \leqq n) \\ = \varphi_{n+1}(X_{i_1} \cap \cdots \cap X_{i_q} \cap X_{n+1}) (i_q \leqq n). \end{cases}$$

The number of the sheets of the covering $\mathcal{B}^n = \{X_i \cap X_{n+1};\ i = 1 \cdots n\}$ is n. From (*) we can regard φ'_{n+1} as an element φ'_n of $Z^{q-1}(K^{p*}_{p.g}(\mathcal{B}^n))$ by using the covering $\mathcal{B}^n$. By the assumption of the induction there exists $\tilde{\varphi}_n \in K^{pq-2}_{p.g}(\mathcal{B}^n)$ such that $\delta_2 \tilde{\varphi}_n = \varphi'_n$. We pull back $\tilde{\varphi}_n$ and we have $\tilde{\varphi}_{n+1} \in K^{pq-1}_{p.g}(\mathcal{N}^{n+1})$.

Q.E.D.

proof of (2) in the Proposition(3.1)

By $H^2(K^{p*}_{p.g}(\mathcal{N}^2)) = 0$, Lemma(3.2) and Lemma(3.3) we have $H^2(K^{p*}_{p.g}(\mathcal{N}^3)) = 0$. By $H^2(K^{p*}_{p.g}(\mathcal{N}^3)) = 0$, Lemma(3.2) and Lemma(3.3) we have $H^2(K^{p*}_{p.g}(\mathcal{N}^4)) = 0$. By using Lemma(3.2),(3.3) repeatedly we have (2).

Q.E.D.

proof of Th.(3.0)

By using Prop.(3.1) and the commutative diagram (*) we have

(a) $H^p(C^*(X,E)_{p.g}) \cong H^p(C^*(\mathcal{U},E_{alg.}))$ for $p \geqq 0$.

By the Serre's vanishing theorem we get

(b) $H^p(C^*(\mathcal{U},E_{alg.})) = 0$ for $p \geqq 1$.

Therefore this completes the proof by (a) and (b).

Q.E.D.

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Hakuki Yamaguchi
Department of Mathematics
Faculty of Science and
Engineering Waseda University
4-170 Nishi-Ohkubo,Shinjuku
Tokyo,Japan